Quantitative Finance Qualifying Exam

May 2019

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

1. **Currency futures**

   The interest rates in the UK $r_{UK} = 0.04$ and US $r_{US} = 0.06$, compounded continuously. The spot price of the UK pound is \$1.60 and the forward price for the UK]pound deliverable in 6-months is \$2.00.

   (a) Does an arbitrage opportunity exist? Show clearly why one is or is not available.

   (b) If there is such an opportunity, describe the trade and show what the riskless profit would be.

2. **Chooser option**

A standard chooser European option is one in which the option holder has the right to decide if the option is a put or a call at some point prior to the expiry of the option. We assume that the strike price K is the same for both the put and call. To simplify the algebra assume the current time is 0, the time at which the choice must be made is τ, and the expiry is T with $0 < \tau < T$. Under the risk neutral measure Q the price of the option is

$$F(0) = e^{-r\tau} E^Q[\max(C(\tau|K,T), P(\tau|K,T)].$$

That is the current price of the chooser is the risk-neutral expected value of the maximum value of a put and a call at the choice point τ. Derive a solution to F(0) above using, as needed, vanilla European puts and calls and cash positions held or borrowed.

3. **Tangent portfolio**

You have one unit of capital available and are a mean-variance optimizer. Let $\boldsymbol{\mu}$ denote the return mean vector, $\boldsymbol{\Sigma}$ the return covariance matrix, r_f the risk-free rate, and $\mathbf{x}$ the asset allocation vector. Assume that assets can be shorted. Also assume that investments can be funded by borrowing cash at the risk-free rate and that unused capital can be invested at the risk-free rate. In this framework, the net cash position is

$$1 - \mathbf{1}^T\mathbf{x} = 1 - \sum_i x_i.$$

(a) Formulate a quadratic program whose solutions represent the mean-variance efficient set of portfolios.

(b) Produce a closed form solution x* that represents the tangent portfolio in the efficient set above.

4. **Power law tails**

A distribution is said to have a power law tail if its survival function has the form:

$$Prob(R > r) = 1 - F(r) = L(r)r^{-\alpha}, \qquad \alpha > 0,$$

where $F(r)$ is the cumulative distribution function of R and $L(r)$ is a slowly varying function such that

$$\lim_{r\to\infty} \frac{L(\lambda r)}{L(r)} = 1, \qquad \text{for any } \lambda > 0.$$

For return distribution with a power law tail, demonstrate mathematically which moments (i.e., $E(R^i)$, $i = 1, 2, 3, \dots$) of R exist depending upon the value of the tail exponent α.

5. **Factor models and mean-variance optimization**

You are given the factor model for the assets whose return vector is $\mathbf{r}(t)$ with factor returns $\mathbf{f}(t)$ and noise term $\boldsymbol{\epsilon}(t)$:

$$\mathbf{r}(t) = \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}(t) + \boldsymbol{\epsilon}(t).$$

The factors are, by construction, orthonormal; i.e., uncorrelated with variance 1. Let the mean factor returns be represented by the vector ϕ.

(a) Express the returns' mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ in terms of the factor model.

(b) Given the mean-variance program below, explain how you can exploit the structure of the factor model to solve the program more efficiently?

$$\min \left\{ \frac{1}{2}\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x} - \lambda\boldsymbol{\mu}^T\mathbf{x} \right\}$$

(c) Assuming there are $n = 500$ assets and $m = 10$ factors, compare the number of parameters needed to compute the covariance matrix under the factor model with one computed directly from the raw returns.

6. **Capital asset pricing model**

You are given three stocks, $i = 1, 2, 3$. Each stocks return can be modeled by the Capital Asset Pricing Model (CAPM):

$$r_i - r_f = \beta_i(r_m - r_f) + \epsilon_i.$$

Let r_i = the return of stock i, r_f = the risk-free rate, β_i = the beta (market exposure) of stock i, r_m = the market return, and ϵ_i = the error term for stock i.

Let $r_f = 0.01$, $(\beta_1, \beta_2, \beta_3) = (0.8, 1.0, 1.2)$, $E(r_m) = \mu_m = 0.08$, $\sigma_m = 0.01$, and $(\sigma_{\epsilon_1}, \sigma_{\epsilon_2}, \sigma_{\epsilon_3}) = (0.08, 0.06, 0.11)$.

(a) What is the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ for mean-variance portfolio problem?

(b) Assuming short positions are permitted and no constraints, derive the closed-form solution of the mean- variance optimization problem to proportionality.

(c) Assuming unit capital (i.e., the allocations sum to 1), apply the proportional solution derived to the problem above to determine the optimal portfolio.

7. **Brownian motion and its first passage time**

Consider a standard Brownian motion W_t with $W_0 = 0$ and the first-passage time $\tau = \{t \geq 0; W_t = m\}$. Show that the density of τ is

$$f_\tau(t) = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, \qquad t \geq 0.$$

8. **CIR model for interest rates**

Consider the CIR model for interest rate r_t

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dW_t,$$

where W_t is a standard Brownian motion with $W_0 = 0$.

(1) Show that

$$r_t = e^{-\beta t}r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\sqrt{r_s}dW_s.$$

(2) Compute the unconditional mean $E(r_t)$

9. **Black-Scholes pricing theory**

Suppose the stock prices S_1 and S_2 follow geometric Brownian motions, $dS_{i,t} = \mu_i S_{i,t} dt + \sigma_i S_{i,t} dW_{i,t}$, $i = 1, 2$, where μ_i and σ_i are the drift and volatility of the process, and $W_{i,t}$ are two independent standard Brownian motions with $W_{i,0} = 0$. Assuming constant interest rate r, perfectly divisible securities, zero dividends and no transaction costs. Consider a European type financial claim that pays $\max(S_{1,T}/S_{2,T} - K, 0)$ at maturity T. (1) Derive the differential equation for the price of the option. (2) What is the price of the financial claim at time t?

10. **ES in ARMA-GARCH model**

Consider the following ARMA(1, 1)-GARCH(1, 1) model for the daily return r_t of an asset:

$$r_t = \theta r_{t-1} + u_t + \psi u_{t-1}, \quad u_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where ϵ_t are independent and identically distributed standard normal random variables, and $\theta, \psi, \alpha, \beta$ satisfy conditions that make r_t stationary. Compute the 99% 2-day expected shortfall of a long position at time t.

11. **Properties of Gumbel copula**

Consider the following Gumbel copula

$$C_\theta^{Gu}(u_1, u_2) = \exp\{-\big[(-\log u_1)^\theta + (-\log u_2)^\theta\big]^{1/\theta}\}, \qquad \theta \geq 1 \tag{1}$$

Show that, when $\theta \to \infty$, $C_\theta^{Gu}(u_1, u_2)$ becomes a 2-d comonotonicity copula.

12. **Tail behavior of AR-GARCH models**

Consider the GARCH(1,1) model

$$y_t = \alpha y_{t-1} + \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where ϵ_t are independent and identically distributed standard normal random variables. Compute the kurtosis of the series $\{y_t\}$ and show it is larger than 3.