APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Fall 2021 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 26, 2021

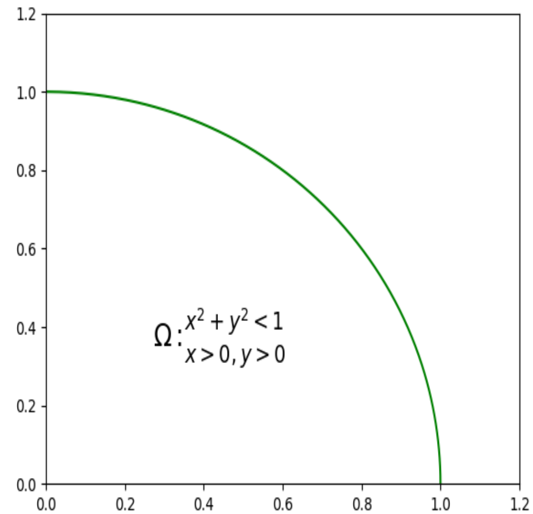
Time: 9:00 AM – 1:00 PM

Detailed Instructions for Zoom Proctored Qual: CAM Area Exam

1. This exam is conducted **via Zoom on May 26, 2021, from 9:00 am to 1:00 pm EDT**.
2. The entire Zoom meeting and chat messages are being recorded.
3. This is a closed book, closed note exam.
4. Hand calculators (or other computing devices) may not be used during the exam.
5. Zoom: You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
6. Audio should be muted, and video must be kept on during the exam.
7. Your computer webcam must fully show your face; your smartphone camera should show your hands and workspace, with the pages of paper being used for the exam.
8. At the very beginning of the exam, during set up, you will be asked to do a brief "environment scan", showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
9. You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the "environment scan" on Zoom.
10. You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication with the proctor(s) via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
11. After you finish the exam, scan all of your pages of work, into a single pdf, and email as an attachment to `xiaolin.li@stonybrook.edu` no later than 5 minutes after completion of the exam (i.e., **by 1:05pm, EDT, on Wed, May 26**).
12. For each question you answer, start at the top of a clean page of paper and label the problem clearly. If the problem has an associated figure (on the exam page), you can write directly on that figure and include it in the scanned solutions you send by email.
13. Scan and submit your answers in a single pdf file. Make sure you scan carefully, so that images are clear, not blurred or truncated.
14. No students are allowed to leave the Zoom meeting until the exam is over.
15. If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 1:00pm, EDT.
16. After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
17. If your answers are not returned by 1:05 pm, EDT, the exam will not be graded, and a score of zero will be given.
18. If you have a question during the exam, then send a Zoom chat message to the host.

A1. Solve Laplace equation $\Delta u = 0$ in the domain Ω shown by the figure with the boundary condition

$$u(x,0) = 0, \quad u(0,y) = 1, \quad u(x,y)|_{x^2+y^2=1} = y. \quad \left(\text{In polar coordinates}: \ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right).$$



A2. The traffic flow follows the conservation law

$$u_t + f(u)_x = 0,$$

where $u > 0$ is the traffic density and $f(u) = u(10 - u)$ is the traffic flux. The initial conditions for (a)-(c) are given as

$$u(x,0) = \begin{cases} u_l & x < -1 \\ u_m & -1 < x < 1 \\ u_r & x > 1 \end{cases}$$

(a). Draw the characteristics and find solution $u(x,t)$ if $u_l = u_m = 2$ and $u_r = 8$.

(b). Draw the characteristics and find solution $u(x,t)$ if $u_l = 8$ and $u_m = u_r = 2$.

(c). Draw the characteristics and find solution $u(x,t)$ if $u_l = 2$, $u_m = 5$ and $u_r = 8$.

(d). Solve the Riemann problem

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

A3. Let $f(z)$ be analytic on the entire complex plane $\mathbb{C}$. Show that $f(z)$ is constant if $|f(z)| \leq \log(1 + |z|)$ for all $z \in \mathbb{C}$.

A4. Find an explicit conformal map from region A_1 to region A_2 where

$$A_1 = \{z \in \mathbb{C},\ \text{Re}(z) > 0,\ \text{Im}(z) > 0,\ |z| < 1\},$$

$$A_2 = \{z \in \mathbb{C},\ \text{Re}(z) > 0,\ \text{Im}(z) > 0\}.$$

A5. Calculate the integral

$$\int_0^\infty \frac{\cos(x)}{x^4 + x^2 + 1} dx.$$

Be sure to include the main details of your calculation.

B6. The kth moment is defined as $\int x^k \, dx$.

a) (4 points) How many quadrature points (a.k.a. nodes) do you need for a Newton-Cotes rule to integrate all kth moments exactly over an interval $[a, b]$ for $k = 0, 1, \ldots, n$? How many quadrature points do you need for a Gaussian quadrature rule?

b) (3 points) Derive a Newton-Cotes rule that can integrate the kth moments $\int_a^b x^k \, dx$ over an interval $[a, b]$ exactly for k up to 3. What is the degree of this rule?

c) (3 points) Explain how to derive a Gaussian quadrature rule to integrate the kth moments exactly up to $k = 3$. Just give the equations for solving for the points and weights; you do not need to solve the equations manually.

B7. For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, consider the methods

$$y_{i+1} = y_i + h\left(\alpha f_i + (1-\alpha) f_{i-1}\right),$$

where f_k denotes $f(t_k, y_k)$ and $\alpha \in \mathbb{R}$, and h is the time step.

a) (5 points) Determine the order of the accuracy of the method as a function of α.

b) (5 points) Consider the special case $f(t, y) = \lambda y$ with $\lambda \in \mathbb{R}$. Find the maximum h such that the method is stable for $\alpha = 1$. Note that h may depend on λ.

B8. Poisson equation and Fourier transform.

a) (4 points) Use the standard second-order centered-difference approximation to discretize the Poisson equation in one dimension with periodic boundary conditions:

$$\begin{aligned} u''(t) &= f(t),\ 0 \le t \le 1 \\ u(0) &= u(1). \end{aligned}$$

Show the resulting linear system for a uniform grid with n intervals. What is the null space of the resulting linear system?

b) (4 points) Show that each column of the Fourier matrix $\boldsymbol{F}_n$ is an eigenvector of the coefficient matrix $\boldsymbol{A}$ in the resulting linear system in (a). Recall that $\{\boldsymbol{F}_n\}_{mk} = \omega_n^{mk}$ for $m = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, n-1$, where $\omega_n = e^{-2\pi i/n}$.

c) (2 points) Suppose n is a power of 2. Describe how to one can utilize FFT to solve the linear system from (a).

B9. Using the discrete Fourier transform, investigate stability of the following implicit scheme for the advection-diffusion equation $v_t + av_x = \nu v_{xx}$

$$u_k^{n+1} + \frac{a\Delta t}{\Delta x}(u_k^{n+1} - u_{k-1}^{n+1}) - \frac{\nu \Delta t}{\Delta x^2}(u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}) = u_k^n.$$

Consider the case of positive and negative velocity a.

B10.

(a) Formulate the Lax-Wendroff theorem on conservative methods for nonlinear conservation laws. Describe limitations of this theorem.

(b) Define the total variation and the TV-stability for a conservative method for nonlinear hyperbolic conservation laws.

(c) Formulate and prove the nonlinear stability theorem that relates TV and TV_T.