# AMS Foundation Exam - Part A, January 19, 2023

Name: ______________________________ ID Num. ____________________

LA: ______ / 30

AC: ______ / 30 Total: ______ / 60

This component of the Foundation Exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with three problems in each. Each question is worth 10 points; answer all **THREE** questions from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Show all work and/or justify your responses.

**Your solutions must be submitted within 5 minutes of the end of the exam. Submission instructions:**

1. Scan your pages, ordered and oriented appropriately, into a single PDF file. Make sure that each problem's solution is clearly labeled.
2. Email the PDF file to Professor Li (`xiaolin.li@stonybrook.edu`), CCing Professor Green (`david.green@stonybrook.edu`). Your email should be titled "AMS Foundation Exam Part A" and does not need to contain any text in its body.
3. Late submissions – those received after 11:05 am EST on January 19, 2023 (as timestamped by the SBU email service) – will not be scored.

**Good Luck!**

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Signature**

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## Section 1: Linear Algebra

1. A tridiagonal matrix is a band matrix that has nonzero elements on and only on the main diagonal, the subdiagonal (the first diagonal below this), and the supradiagonal (the first diagonal above the main diagonal). For example, the following is a $4 \times 4$ tridiagonal matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

   (a). Calculate $\det(A)$.

   (b). Find $A^{-1}$.

   (c). For an $N \times N$ tridiagonal matrix, how many arithmetic operations are needed to calculate its determinant and find the inverse in term of $N$?

   (d). Find an orthonormal basis of the column space.

2. Find a basis for the **orthogonal complement** of the subspace in $\mathcal{R}^4$ spanned by the following vectors

$$v_1 = \begin{pmatrix} 1 \\ 4 \\ 5 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 3 \\ 2 \\ 2 \end{pmatrix}.$$

3. Given the following matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

(a). Find all the eigenvalues and eigenvectors.

(b). Find $P$ and $P^{-1}$ such that $P^{-1}AP = \Lambda$ is a diagonal matrix.

(c). Given any vector $x \in \mathcal{R}^3$ with nonzero norm ($||x|| \neq 0$), analyze and evaluate the following limit

$$\lim_{n\to\infty} \frac{||A^{n+1}x||}{||A^n x||}.$$

## Section 2: Advanced Calculus

1. Use the Cauchy-Schwarz inequality to show the following

   (a).
$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \geq n^2$$
   where $a_i, \ i = 1, 2, \cdots, n$ are real and positive numbers.

   (b).
$$\left(\int_a^b |f(x)| dx\right)\left(\int_a^b \frac{1}{|f(x)|} dx\right) \geq (b-a)^2$$
   where $f(x) \neq 0$, for $x \in [a, b]$.

2. Given an ellipsoid with uniform mass density $\sigma$

$$V : \{(x, y, z) \in R^3, x^2 + y^2 + 4z^2 \leq 1\}.$$

(a). Calculate the total volume of the ellipsoid.

(b). Calculate the moment of inertia when rotation about the $z$-axis. The moment of inertia is

$$I = \iiint_V \sigma d^2 dV$$

where $d$ is the distance betwee the point in the integration and the rotation axis (in this case: $d = \sqrt{x^2 + y^2}$).

(c). Calculate the moment of inertia when rotation about the axis in $z$-direction, but through the point $(1, 0, 0)$ (in this case: $d = \sqrt{(x-1)^2 + y^2}$).

3. Calculate the following, if exist

(a).

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy^2)}{x^2y}.$$

(b).

$$\lim_{x\to1}\frac{1-4\sin^2\frac{\pi x}{6}}{1-x^2}$$

(c).

$$\int_0^1 x^a\,(\ln x)^m\,dx,$$

where $a > -1$ and $m$ is a nonnegative integer.