

**APPLIED MATHEMATICS and STATISTICS  
DOCTORAL QUALIFYING EXAMINATION  
in COMPUTATIONAL APPLIED MATHEMATICS**

**Summer 2018 (May)**

**(CLOSED BOOK EXAM)**

**This is a two part exam.  
In part A, solve 4 out of 5 problems for full credit.  
In part B, solve 4 out of 5 problems for full credit.**

Indicate below which problems you have attempted by circling the appropriate numbers:

<b>Part A:</b>	1	2	3	4	5
<b>Part B:</b>	6	7	8	9	10

**NAME:** \_\_\_\_\_

**STUDENT ID:** \_\_\_\_\_

**SIGNATURE:** \_\_\_\_\_

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 23, 2018

Time: 9:00 AM – 1:00 PM

**A1.** Consider the two dimensional Laplace equation in a circular disk

$$\Delta u = 0, \quad |x| < R, \quad u|_{|x|=R} = \sin \phi.$$

- (a). Solve the boundary value problem using separation of variable method.
- (b). Solve the boundary value problem using Green's function method.
- (c). If the equation is changed to Poisson's equation

$$\Delta u = f(x),$$

find the solution.

In polar coordinate

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2}$$

**A2.** The heat equation in  $R$  space

$$u_t = u_{xx}, \quad x \in (-\infty, \infty), \quad t > 0$$

with initial condition  $u(x, 0) = g(x)$  is solved with exact solution

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} g(y) e^{-\frac{(x-y)^2}{4t}} dy$$

(a). Draw the initial condition

$$u(x, 0) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x+2)^2}{4}} - \frac{1}{\sqrt{4\pi}} e^{-\frac{(x-2)^2}{4}}.$$

(b). Find the solution for  $t > 0$ .

(c). Sketch the trend of the solution as  $t$  increases.

(d). Find the approximate maximum and minimum of the solution in the solution domain.

**A3.** The flood wave in a river follows the conservation law

$$A_t + (A^{3/2})_x = 0,$$

where  $A(x, t)$  is the cross sectional area of the water at the location  $x$  and time  $t$ . A sudden heavy rain creates a flood with river cross sectional area along its path as follows (set it as  $t = 0$ )

$$A(x, 0) = \begin{cases} 1 & x \leq 0 \\ 4 & 0 < x \leq 10 \\ 1 & x > 10 \end{cases}.$$

- (a). Find and draw the characteristics of the equation.
- (b). Find the solution of cross sectional area along the river at  $t = 1$ .
- (c). Assume a town is located at  $x = 31$ , when will the flood crest reach it?

**A4.** Let  $f(z)$  be an entire function satisfying  $|f(z)| \leq |z|^2$  for all  $z \in \mathbb{C}$ . Show that  $f(z) = az^2$  for some  $|a| \leq 1$ .

**A5.** Find an explicit conformal map from the unit disc to the infinite strip  $|\operatorname{Im} z| < 1$ .

**B6.**

- (a). Derive the parameters of Gauss quadrature with three points

$$\int_{-1}^1 f(x)dx \approx C_1 f(\xi_1) + C_2 f(\xi_2) + C_3 f(\xi_3)$$

such that the integral is exact up to  $x^5$ .

- (b). Use the derived parameter in (a) to calculate

$$I = \int_a^b g(x)dx.$$

- (c). If we divide  $[a, b]$  into  $2m$  segments and apply this Gauss quadrature to each pair of segments, count the total number of arithmetic operations (addition, subtraction, multiplication and division) and total number of function evaluations for the calculation of the integral in terms of  $m$ .
- (d). What is the order of the local truncation error and what is the order of the global truncation error?

**B7.** To find the root of  $f(x) = 0$ , we start from Newton's interpolation formula on two points  $x_n$  and  $x_{n-1}$ .

(a). Write down the interpolation function  $p_1(x)$  and the exact  $f(x)$  through Newton divided differences.

(b). Show that the secant method is equivalent to

$$p_1(x_{n+1}) = 0.$$

(c). Use the fact that  $f(\xi) = 0$  and  $p_2(x_{n+1}) = 0$ , to derive

$$e_{n+1} = x_{n+1} - \xi = A(x_n - \xi)(x_{n-1} - \xi) = Ae_n e_{n-1}.$$

What is  $A$ ?

(d). Convert the  $e_{n+1}$  to the form

$$e_{n+1} = Ce_n^r.$$

Find the value of  $r$  and the relation between  $C$  and  $A$ .



**B8.** Consider a 1D linear advection equation  $v_t + av_x = 0$ .

- (a). Write the Lax-Friedrichs and the Lax-Wendroff methods for this equation in the conservative form and find the corresponding fluxes.
- (b). Derive a flux-limiter method using these fluxes. No need to explicitly specify the limiter function.

**B9.** Derive a corrected (corner-transport) upwind method for the linear advection equation in 2D space

$$\frac{\partial v(x, y, t)}{\partial t} + a \frac{\partial v(x, y, t)}{\partial x} + b \frac{\partial v(x, y, t)}{\partial y} = 0.$$

**B10.** Using the discrete Fourier transform, perform stability analysis of the leapfrog scheme for the 1D diffusion equation  $v_t = \nu v_{xx}$ .