Bayesian Level Set Inversion

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Stony Brook University,
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EPSRC

Outline

Outline

Linear Problem

H. W. Engl, M. Hanke and A. Neubauer
Regularization of Inverse Problems.
Kluwer (1994)

Forward Problem

Let $K \in \mathcal{L}(X, \mathbb{R}^J)$ for some Banach space X. Given $u \in X$

$$y = Ku.$$

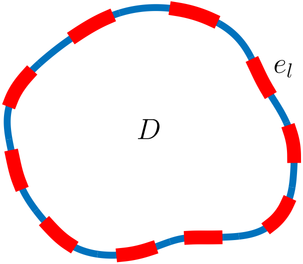
Let $\eta \in \mathbb{R}^J$ be a realisation of an observational noise.

Inverse Problem

Given prior information on $u \in X$, and given $y \in \mathbb{R}^J$, find u :

$$y = Ku + \eta.$$

Electrical Impedance Tomography



- Apply currents I_ℓ on e_ℓ, $\ell = 1, \dots, L$.
- Induces voltages Θ_ℓ on e_ℓ, $\ell = 1, \dots, L$.
- Input is (σ, I), output is (θ, Θ).
- We have an Ohm's law $\Theta = R(\sigma)I$.

$$
\begin{cases}
-\nabla \cdot (\sigma(x)\nabla\theta(x)) = 0 & x \in D \\
\displaystyle\int_{e_\ell} \sigma \frac{\partial\theta}{\partial\nu}\, \mathrm{d}S = I_\ell & \ell = 1, \dots, L \\
\sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = 0 & x \in \partial D \setminus \bigcup_{\ell=1}^{L} e_\ell \\
\theta(x) + z_\ell \sigma(x)\dfrac{\partial\theta}{\partial\nu}(x) = \Theta_\ell & x \in e_\ell, \ell = 1, \dots, L
\end{cases}
\quad \text{(PDE)}
$$

Electrical Impedance Tomography

M. M. Dunlop and A. M. Stuart
The Bayesian Formulation of EIT.
arXiv:1509.03136
Inverse Problems and Imaging, Submitted, 2015.

Forward Problem

- Let $X \subseteq L^\infty(D)$, and denote $X^+ := \{u \in X : \mathrm{essinf}_{x\in D}\, u > 0\}$.
- Given $u \in X, F : X \to X^+$ and $\sigma = F(u)$, find $(\theta, \Theta) \in H^1(D) \times \mathbb{R}^L$ solving (PDE).
- This gives $\Theta = R(F(u))I$.

Let $\eta \in \mathbb{R}^L$ be a realisation of an observational noise.

Inverse Problem

Given prior information on u, and given currents I and $y \in \mathbb{R}^L$, find u :

$$y = R(F(u))I + \eta.$$

General Structure

A. M. Stuart
Inverse problems: a Bayesian approach.
Acta Numerica **19** *(2010)*

Forward Problem

Let X, Y be separable Banach spaces, and let $\mathcal{G} : X \to Y$ be a measurable mapping. Given $u \in X$,

$$y = \mathcal{G}(u).$$

Let $\eta \in Y$ be a realisation of an observational noise.

Inverse Problem

Given prior information on u, and given $y \in Y$, find u:

$$y = \mathcal{G}(u) + \eta.$$

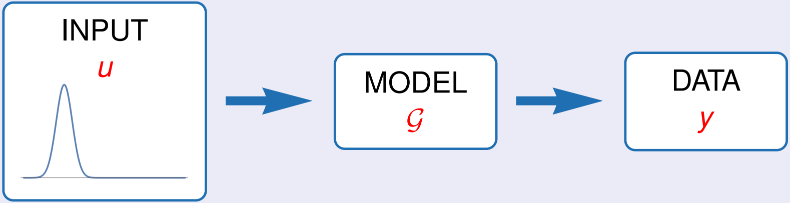
Outline

Bayesian Inversion: The Idea

The Idea: Words

Problem is under-determined; data is noisy. Probability delivers missing information and accounts for observational noise.

The Idea: Picture



Bayesian Inversion: The Idea

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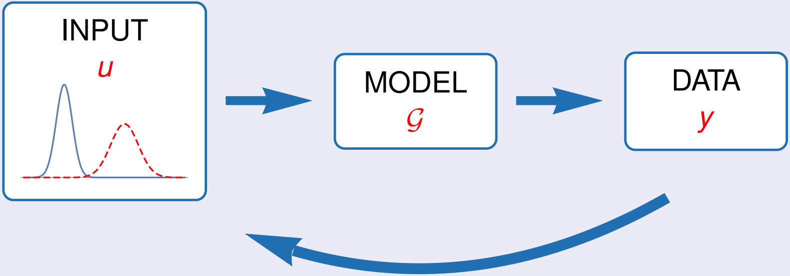

Bayes' Formula

Prior

Probabilistic information about u before data is collected:

$$\mu_0(du)$$

Likelihood

Since $y = \mathcal{G}(u) + \eta$, if $\eta \sim N(0, \Gamma)$, then $y|u \sim N(\mathcal{G}(u), \Gamma)$. The model-data misfit Φ is the negative log-likelihood:

$$\mathbb{P}(y|u) \propto \exp\big(-\Phi(u;y)\big), \quad \Phi(u;y) = \frac{1}{2}\Big|\Gamma^{-1/2}(y - \mathcal{G}(u))\Big|^2.$$

Posterior

Probabilistic information about u after data is collected:

$$\mu^y(du) \propto \exp\big(-\Phi(u;y)\big)\mu_0(du).$$

Well-posedness

A. M. Stuart
Inverse problems: a Bayesian approach.
Acta Numerica **19** *(2010)*

$$L^2_\nu(X;S) = \{f : X \to S : \mathbb{E}^\nu \|f(u)\|^2_S < \infty\}.$$

Theorem

Assume that:

- $u \in X$ μ_0−a.s. ;
- $\mathcal{G} \in C(X, \mathbb{R}^J)$;
- $\mathcal{G} \in L^2_{\mu_0}(X; \mathbb{R}^J)$.

Then $y \mapsto \mu^y(du)$ is Lipschitz in the Hellinger metric. Furthermore, if S is a separable Banach space and $f \in L^2_{\mu_0}(X; S)$, then

$$\left\|\mathbb{E}^{\mu^{y_1}} f(u) - \mathbb{E}^{\mu^{y_2}} f(u)\right\|_S \leq C|y_1 - y_2|.$$

Probing the Posterior

We wish to get information about the structure of the posterior probability μ^y on unknown function u given data y. Possibilities:

- Best approximation by a Dirac: MAP/Tikhonov

M. Dashti, K. J. H. Law, A. M. Stuart and J. Voss
MAP estimators and their consistency in Bayesian nonparametric inverse problems.
Inverse Problems **29** *(2013)*

- Best approximation by a Gaussian: variational/ML

F. J. Pinski, G. Simpson, A. M. Stuart and H. Weber
Kullback-Leibler approximation for probability measures on infinite dimensional spaces.
SIAM J. Math. Analysis (to appear)

- Best approximation by many Diracs: sampling/MCMC

S. L. Cotter, G. O. Roberts, A. M. Stuart, and D. White.
MCMC methods for functions: modifying old algorithms to make them faster.
Statistical Science **28** *(2013)*

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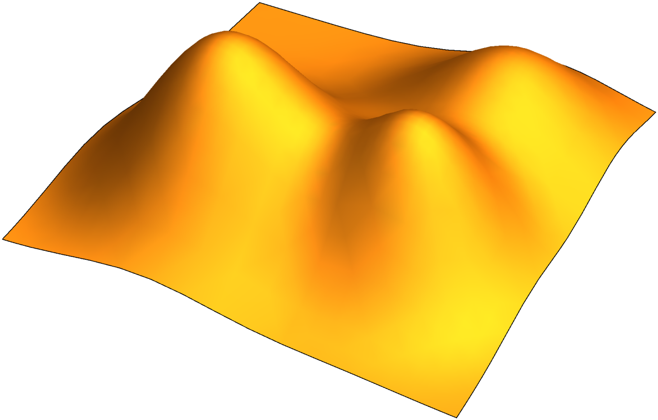
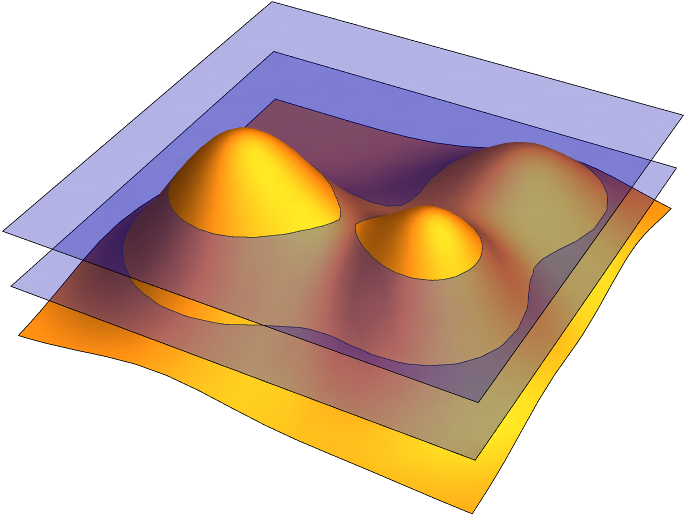
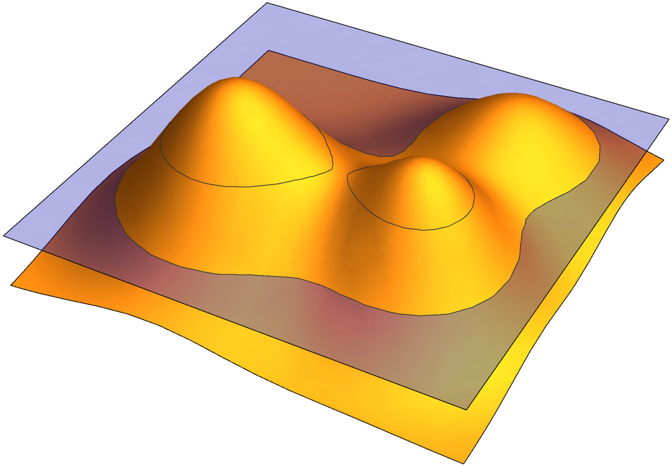
The Level Set Approach to Inverse Problems



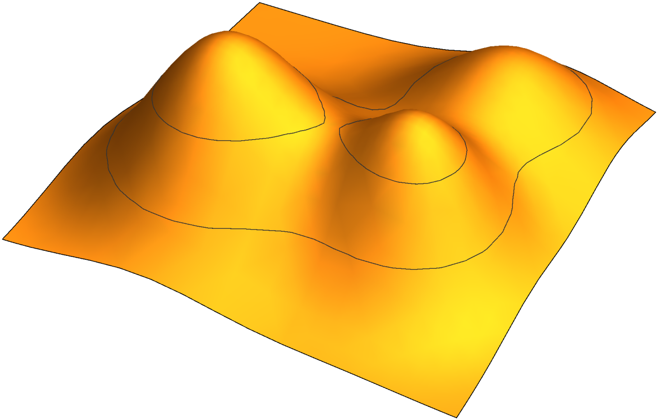
The Level Set Approach to Inverse Problems



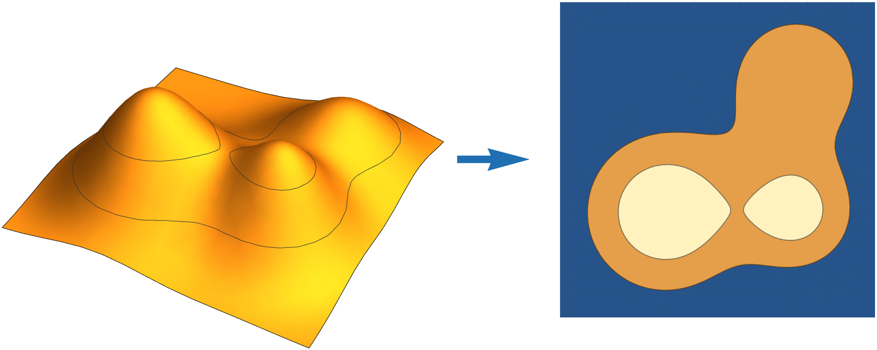
The Level Set Approach to Inverse Problems



The Level Set Approach to Inverse Problems



The Level Set Approach to Inverse Problems



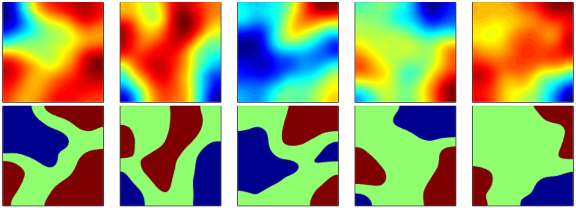
- Recovery of a piecewise constant field now becomes recovery of a continuous field.

Level Set Inversion: The Level Set Map

M. A. Iglesias, Y. Lu and A. M. Stuart
A level-set approach to Bayesian geometric inverse problems
arXiv:1504.00313
Interfaces and Free Boundaries, Submitted, 2015.

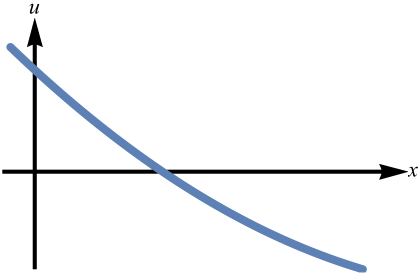
Piecewise constant conductivity σ (EIT example) defined through thresholding a level set function u:

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i \, \mathbb{I}_{\{c_{i-1} < u \le c_i\}}(x).$$

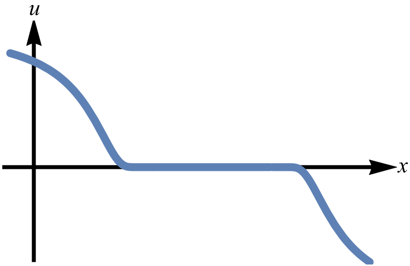


$\sigma = F(u)$, u is now the unknown.

Level Set Inversion: A Continuity Issue



$F(\cdot)$ is continuous at u



$F(\cdot)$ is discontinuous at u

$$\sigma = F(u) := \sigma^{+}\mathbb{1}_{\{u \geq 0\}}(x) + \sigma^{-}\mathbb{1}_{\{u<0\}}(x)$$

Level Set Inversion: Well-posedness

M. Iglesias, Y. Lu and A. M. Stuart
A level-set approach to Bayesian geometric inverse problems. (above)

M. M. Dunlop and A. M. Stuart
The Bayesian formulation of EIT: analysis and algorithms. (above)

Level Set Measurement Set-Up

- $F : X \to Z$, $X = C(D; \mathbb{R}), Z = L^\infty(D; R)$; level-set map.
- $G : Z \to \mathcal{H}$, $\mathcal{H}$ Hilbert space; PDE solve/linear map.
- $\mathcal{O} : \mathcal{H} \to \mathbb{R}^J$; linear functionals of solution.

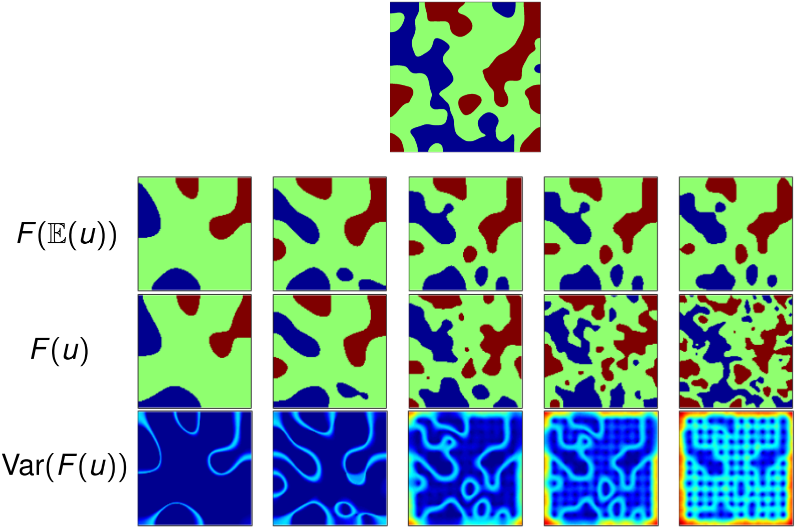
Theorem

Assume that $\mathcal{G} := \mathcal{O} \circ G \circ F : X \to \mathbb{R}^J$ and, for Gaussian prior μ_0, $u \in X$ with probability 1. Then, for the linear and EIT examples, $y \mapsto \mu^y(du)$ is Lipschitz in the Hellinger metric. Furthermore, if S is a separable Banach space and $f \in L^2_{\mu_0}(X; S)$, then

$$\left\| \mathbb{E}^{\mu^{y_1}} f(u) - \mathbb{E}^{\mu^{y_2}} f(u) \right\|_S \leq C|y_1 - y_2|.$$

Outline

Length-scale is Important



A Family of Prior Distributions

- Whittle-Matérn distributions allow for control over sample regularity and length scale.
- These are stationary Gaussian distributions with covariance function

$$c_{\nu,\ell}(x,y) = \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{|x-y|}{\ell}\right)^{\nu} K_{\nu}\left(\frac{|x-y|}{\ell}\right).$$

- Special cases are exponential ($\nu = 1/2$) and Gaussian ($\nu \to \infty$) covariance functions.
- Ignoring boundary conditions, the covariance operator $\mathcal{D}_{\nu,\ell}$ corresponding to the covariance function $c_{\nu,\ell}$ is given by

$$\mathcal{D}_{\nu,\ell} = \beta\ell^{d}(I - \ell^{2}\Delta)^{-\nu-d/2}.$$

We Need to Re-scale

- The factor ℓ^d leads to problems when finding algorithms that are robust with respect to mesh refinement (lack of absolute continuity).
- Hence re-scale the covariances as $\mathcal{C}_{\nu,\ell} = (\ell^{-2} I - \Delta)^{-\nu - d/2}$.
- For $u \sim N(m_0, \mathcal{C}_{\nu,\ell})$, we have $\mathbb{E}\|u - m_0\|^2 \propto \ell^{2\nu}$.
- To counter this, scale levels c_i with ℓ as well:

$$c_i(\ell) = m_0 + \ell^\nu (c_i - m_0).$$

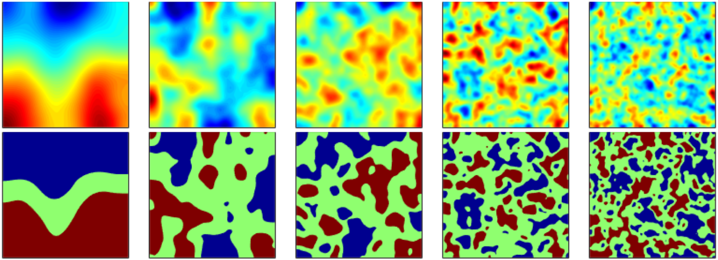
- This means we must explicitly pass the length scale parameter ℓ to the level set map.

The New Level Set Map

M. M. Dunlop, M. A. Iglesias and A. M. Stuart
Hierarchical Bayesian Level Set Inversion
In preparation

Let $X = C^0(D)$ and $Z = L^p(D)$. $F : X \times \mathbb{R}^+ \to Z$ is defined by

$$F(u,\ell) = \sum_{k=1}^{K} \sigma_k \mathbb{1}_{\{c_{k-1}(\ell) \le u < c_k(\ell)\}}.$$



A Sampling Algorithm

We can sample the posterior $\mu^y(\mathrm{d}u, \mathrm{d}\ell)$ using a Metropolis-within-Gibbs MCMC method:

Algorithm

1. *Set $k = 0$ and pick initial state $(u^{(0)}, \ell^{(0)}) \in X \times \mathbb{R}^+$.*
2. *Update $u^{(k+1)} \sim u|(\ell^{(k)}, y)$ using a dimension robust MCMC.*
3. *Update $\ell^{(k+1)} \sim \ell|(u^{(k+1)}, y)$ using a scalar sampling algorithm.*
4. *$k \to k + 1$ and return to 2.*

Step 3 above requires knowledge of the conditional distribution $\pi^{u,y}$ of $\ell|(u, y)$. The absolute continuity of the family $\{\mu_0^\ell\}_{\ell>0}$ allows us to write down an expression for this.

Outline

Summary

- Overview of the recent development of a theoretical and computational framework for infinite dimensional Bayesian inversion.
    1. Probabilistic well-posedness.
    2. Leads to new algorithms (defined on Banach space).
    3. Mesh-indepedent convergence rates for MCMC.
- A Bayesian level set method overcomes some challenges with classical level set methods.
    1. Probabilistic well-posedness follows from the general theory.
    2. Algorithms which update the level set implicitly via MCMC methods on level set function – no explicit velocity field required for level set interface.
- A hierarchical approach improves the effectiveness of the level set method.
    1. Relies on a family of equivalent Gaussian measures parameterised by the length scale of their samples.
    2. Variation of sample amplitude compensated for by passing the length scale parameter to level set map.

References

S. L. Cotter, G. O. Roberts, A. M. Stuart, and D. White.
MCMC methods for functions: modifying old algorithms to make them faster.
Statistical Science, **28**(2013), 424-446. arXiv:1202.0709.

M. Dashti, K. J. H. Law, A. M. Stuart and J.Voss.
MAP estimators and posterior consistency in Bayesian nonparametric inverse problems.
Inverse Problems **29***(2013), 095017.*

M. Dashti and A. M. Stuart
The Bayesian approach to inverse problems.
Handbook of Uncertainty Quantification
Editors: R. Ghanem, D.Higdon and H. Owhadi, Springer, 2017.
arXiv:1302.6989

M. M. Dunlop and A. M. Stuart
Bayesian formulation of EIT.
arXiv:1509.03136
Inverse Problems and Imaging, Submitted, 2015.

M. M. Dunlop, M. A. Iglesias and A. M. Stuart
Hierarchical Bayesian Inversion.
In Preparation, 2015.

References

M. Hairer, A. M. Stuart and S. Vollmer
Spectral gaps for Metropolis Hastings algorithms in infinite dimensions
Ann. App. Prob. **24** *(2014)*

M. A. Iglesias, Y. Lu and A. M. Stuart
A level-set approach to Bayesian geometric inverse problems.
arXiv:1504.00313
Interfaces and Free Boundaries, Submitted, 2015.

F. Santosa
A level-set approach for inverse problems involving obstacles.
ESAIM: Control, Optimisation and Calculus of Variations **1**(1996)

A. M. Stuart
Inverse problems: a Bayesian perspective.
Acta Numerica, 19(2010) 451–559.
http://homepages.warwick.ac.uk/ masdr/BOOKCHAPTERS/stuart15c.pdf